Tutorial 1

1. Let V be an inner product space and $T \in \mathcal{L}(V)$ be such that for all $u, v \in V$,

$$\langle T(u), T(v) \rangle = \langle u, v \rangle$$

   (a) What can you say about T?

   (b) If $\lambda \in \mathbb{F}$ is an eigenvalue of T, what can you say about λ?

2. Let V be an inner product space and $S, T \in \mathcal{L}(V)$ be such that for all $u, v \in V$,

$$\langle S(u), v \rangle = \langle T(u), v \rangle$$

   Show that $S = T$.

3. Let V be an inner product space. Show that for all $u, v \in V$,

$$\|u\| - \|v\| \leq \|u - v\|$$

4. (a) Let V be a **complex** inner product space and $T \in \mathcal{L}(V)$ be such that for all $u, v \in V$,

$$\langle T(u), v \rangle + \langle T(v), u \rangle = 0 \tag{1}$$

   What can you say about T?

   (b) Now consider $V = \mathbb{R}^2$ with the standard inner product. Show there exists $T \in \mathcal{L}(V)$ such that Eq. (1) holds for all $u, v \in V$ but T does not have the property found in (a).

5. (6.A.22) Let $a_1, \ldots, a_n \in \mathbb{R}$. Show that

$$\left(\frac{a_1 + \cdots + a_n}{n} \right)^2 \leq \frac{a_1^2 + \cdots + a_n^2}{n}$$

   When do we have equality?

6. (Titu's Lemma) Let $a_1, \ldots, a_n, b_1, \ldots, b_n$ be positive real numbers. Show that

$$\frac{(a_1 + \cdots + a_n)^2}{b_1 + \cdots + b_n} \leq \frac{a_1^2}{b_1} + \cdots + \frac{a_n^2}{b_n}$$

   When do we have equality?

7. (6.A.6) Let V be an inner product space and let $u, v \in V$. Show that $\langle u, v \rangle = 0$ if and only if $\|u\| \leq \|u + av\|$ for all $a \in \mathbb{F}$.